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# Relativistic phase space: dimensional recurrences

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## Abstract

We derive recurrence relations between phase space expressions in different dimensions by confining some of the coordinates to tori or spheres of radius  $R$  and taking the limit as  $R \rightarrow \infty$ . These relations take the form of mass integrals, associated with extraneous momenta (relative to the lower dimension), and produce the result in the higher dimension.

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## 1. Establishing the recurrences

Formulations of field theories in higher dimensions are now quite commonplace, with 10 to 12 dimensions featuring prominently, especially in the context of M-theory. Mostly the coordinates which are additional to the usual four spacetime ones are either very small or the fields possess a severely damped behaviour as one moves away from the 3-brane. Thus the extra coordinates are characterized by one or more length scales  $R$ , which are generally miniscule or, if larger, only affect gravity. At the other extreme one can contemplate the  $R$  as being huge; indeed the method of ‘box normalization’ with a large  $R$  has a venerable pedigree and allows us to describe vacuum diagrams or compute properties per unit volume when they so depend. The same method also permits us to make sense of quantities like the volume of spacetime or  $\delta^4(0)$  in the limit  $R \rightarrow \infty$ .

One of the primary objects of interest in these higher dimensions  $D \equiv 2\ell$  is the behaviour of relativistic  $N$ -body phase space  $\rho_N^D$  since it primarily governs the magnitudes of transmutations, ignoring amplitude modulation. Based on earlier coordinate space methods [1], this behaviour has been recently studied [2] and codified [3]; a summary of how the  $N$ -body result in *fixed*, flat  $D$  space can be evaluated by means of Almgren recurrences [4]—mass integrals over smaller  $N$  but with the *same*  $D$ —was given in [2]. In this paper we establish relations between  $\rho$  having the same  $N$ , but different  $D$ , which are quite interesting in their own right. They may well have some use in the context of recent developments in string or M-theory or  $p$ -brane physics; for such theories possess a set of length scales  $R_i$  (or parameters

arising in the extended metric) which serve to constrain the motion of particles to subspaces or ‘branes’ of lower dimension. Each theory produces its own particle spectrum whose spacing is determined by the  $R_i$ . As the limit  $R_i \rightarrow \infty$  is taken, we may anticipate that the particles freely explore the higher-dimensional space, and the corresponding phase space at a given energy reduces to the relativistic phase space for the entire ‘bulk’.

The strategy for deriving such recurrence relations in the flat space limit is quite simple. We just confine one or several of the  $D$ -dimensional coordinates to a torus or sphere of radius  $R$  and take the limit  $R \rightarrow \infty$  at the very end, in much the same way that box normalization is handled. The act of confinement creates a series of discrete modes in the restricted coordinates and phase space must be summed over the various modes, subject to mass bounds. By choosing the topology of the extra space to be spherical or toroidal, the masses of the modes are easy to work out. (Had we been considering more complicated topologies, the mass spectra would have been much harder to calculate although we still expect the  $R_i \rightarrow \infty$  limit to yield results coinciding with our choices.) Because the contributions to the masses from the confined coordinates are inversely proportional to  $R$ , the summation reduces to a mass integral in the limit of enormous  $R$ : hence the result  $\rho_N^D$  in higher  $D$  is expressible as a set of mass integrals of  $\rho$  over lower  $D$ , but with the same  $N$ ; one can readily understand this as the effect of integrating over extra momentum components relative to those in the lower dimension. The forms of such connections are rather intriguing and some are not at all obvious; in fact, for larger  $N$  they are quite intricate. We know, of course, in advance that they must work out somehow; the interest is in the ‘somehow’, not why they do so. For smaller values of  $N$  we are able to check their validity, but verifying them for  $N > 3$  is a daunting task in many cases.

In the following section we suppose that one or several of the coordinates are confined to a torus, so the recurrence relation is effectively that between  $\rho^D$  and  $\rho^{D-1}$ . Its nature is simple since there is only one extra coordinate to contend with, so we are dealing with a one-dimensional sum or integral in the continuum limit. We show how the recurrence pans out for few-body processes. The next two sections deal with the case when there are several extra coordinates (confined to a sphere) and here we encounter a multidimensional summation or integration, which is nontrivial. When  $N = 2$  we demonstrate how the relations work out for any number of extra coordinates, but for  $N = 3$  we have only succeeded in following through the connection when the dimension difference equals two or more, though no doubt it must apply to any  $N$  value.

## 2. Relations between $D$ and $D - 1$ phase space

Let us begin by specifying our notation. Our metric is  $+, -, -, -, \dots$  with a total of  $D$  coordinates. The  $N$ -body phase space integral in *flat* space is defined by

$$\rho_N^D = \left( \prod_{i=1}^N \int d\Omega_{p_i} \right) (2\pi)^D \delta^D \left( p - \sum_{i=1}^N p_i \right) \quad (1)$$

where  $d\Omega_{p_i} \equiv \theta(p_{i0}) \delta(p_i^2 - m_i^2) d^D p_i / (2\pi)^{D-1}$ . We separate spacetime coordinates  $x$  into  $(D - n)$  coordinates called  $\mathbf{x}$  and extra ones labelled  $\vec{y}$ ,  $n$  of which are independent; likewise for the conjugate momenta. (For the purpose of this section  $n$  equals 1, but we shall consider other  $n$  values later on.) The general aim of the exercise is to establish a connection between  $\rho_N^D$  and  $\rho_N^{D-n}$  and see how that works out analytically because it is nontrivial for large  $N$  or  $n$ .

The first step is to confine (periodic)  $y$  to the circumference of a circle of radius  $R$ . Thus the space is considered to have the direct product topology  $M^{D-1}(x) \otimes T^1(y)$ . Then Fourier expand a (real) field  $\Phi$  into modes in the standard way so as to fix the normalization correctly:

$$\Phi(\mathbf{x}, y) = \sum_{k=-\infty}^{\infty} \phi_k(\mathbf{x}) \exp(iky/R) / \sqrt{2\pi R} \quad \phi_k^*(\mathbf{x}) = \phi_{-k}(\mathbf{x}). \quad (2)$$

Being complex in general,  $\phi_k^*$  can be regarded as the antiparticle field to  $\phi_k$ , where  $k$  is positive say.  $y$  must run from  $-\pi R$  to  $\pi R$ , to ensure that, upon integration over  $y$ , the free action takes its proper form,

$$S_{\text{free}} = \frac{1}{2} \int d^D x [(\partial_x \Phi)^2 - m^2 \Phi^2] = \sum_k \frac{1}{2} \int d^{D-1} \mathbf{x} [(\partial_x \phi_k^*)(\partial_x \phi_k) - m_k^2 \phi_k^* \phi_k] \quad (3)$$

where  $m_k^2 \equiv m^2 + (k/R)^2$  corresponds to the mode  $k$  mass squared. Note that if one restricts the sum to positive  $k$  (because of repetition) the factor of  $1/2$  disappears and one gets the right normalization for a complex field. A zero-mode field which is of course real and  $y$ -independent is given by  $\Sigma(\mathbf{x}) = \sigma(\mathbf{x}) / \sqrt{2\pi R}$ .

To determine the phase space integral we consider the point interaction between a heavy field  $\Sigma$  (with mass  $m_0$ ) decaying into  $N$  distinguishable fields which carry their own distinct quantum numbers, so  $\Sigma$  matches all of them. (One may also consider the case where some of the final particles are identical, but that just serves to introduce symmetry factors which must be taken into account and adds little to the discussion.) Write the effective interaction as  $\mathcal{L} = \phi_1 \dots \phi_N \Sigma$ , so that integration over the ‘extra’ coordinate produces

$$\int \mathcal{L}(\mathbf{x}, y) dy = \frac{\sigma}{(2\pi R)^{(N-1)/2}} \left( \prod_{i=1}^N \phi_{ik_i} \right) \delta_{\sum_i k_i, 0}$$

with the sum taken over positive and negative  $k$  values. Thus we deduce the effective coupling of  $\sigma$  with the various  $\phi$  in the lower dimension to be  $g_{k_1 \dots k_N} = \delta_{\sum_i k_i, 0} / (2\pi R)^{(N-1)/2}$ . This means that the higher-dimensional phase space can be written in the more explicit form,

$$\rho_{m_0 \rightarrow m_1 + \dots + m_N}^D = \sum_{k_i} \frac{\delta_{\sum_i k_i, 0}}{(2\pi R)^{N-1}} \rho_{m_0 \rightarrow m_{1k_1} + \dots + m_{Nk_N}}^{D-1} \quad (4)$$

subject to energy–momentum conservation of course, which thus provides upper bounds on the magnitudes of the running  $k$ -values.

The second step is to take the limit as  $(R, k) \rightarrow \infty$  and let  $\mu_i = k_i/R$ . We see that the connection (4) reduces to the continuous version,

$$\rho_{m_0 \rightarrow m_1 + \dots + m_N}^D = \left( \prod_i \int \frac{d\mu_i}{2\pi} \right) 2\pi \delta \left( \sum_i \mu_i \right) \rho_{m_0 \rightarrow m_{1\mu_1} + \dots + m_{N\mu_N}}^{D-1} \quad m_{i\mu}^2 \equiv m_i^2 + \mu^2 \quad (5)$$

where again the range of  $\mu$ -values is restricted by the condition  $m_0 \geq m_{1\mu_1} + \dots + m_{N\mu_N}$ . This last form is quite readily understood as a consequence of writing the mass shell condition,  $0 = p^2 - m^2 = \mathbf{p}^2 - (m^2 + \mu^2)$ , where  $\mu$  stands for the last momentum component  $p_y$ . The phase space integral over  $p_y$  then produces the delta function  $\delta(\sum_i \mu_i)$ , because the initial zero-mode has no dependence on  $p_y$ .

Equation (5) is the recurrence relation we were seeking, so now let us see how it works out for those cases which we can tackle explicitly. Start with the easiest case,  $N = 2$ , where we know that

$$\rho_{m_0 \rightarrow m_1 + m_2}^{2\ell} = \frac{\pi^{1-\ell} \Gamma(\ell - 1) (\lambda(m_0^2, m_1^2, m_2^2))^{2\ell-3}}{2^{2\ell-1} m_0^{2\ell-2} \Gamma(2\ell - 2)} \quad (6)$$

involving the Källén function,  $\lambda(a, b, c) \equiv \sqrt{a^2 + b^2 + c^2 - 2ab - 2bc - 2ca}$ . The recurrence relation (5) reduces to the prediction that

$$\rho_{m_0 \rightarrow m_1+m_2}^{2\ell} = \frac{1}{2\pi} \int d\mu \rho_{m_0 \rightarrow m_{1\mu}+m_{2\mu}}^{2\ell-1} \theta(m_0 - m_{1\mu} - m_{2\mu}) \tag{7}$$

and its verification relies upon the observation that

$$\lambda^2(m_0^2, m_{1\mu}^2, m_{2\mu}^2) = \lambda^2(m_0^2, m_1^2, m_2^2) - 4m_0^2\mu^2$$

plus the basic integral ( $M^2 \equiv \lambda(m_0^2, m_1^2, m_2^2)/2m_0$  and  $r \equiv 2\ell - 3$  below)

$$\begin{aligned} \int d\mu \theta(m_0 - m_{1\mu} - m_{2\mu}) \lambda^r(m_0^2, m_{1\mu}^2, m_{2\mu}^2) &= \int_0^{M^2} \frac{d\mu^2}{\sqrt{\mu^2}} (2m_0\sqrt{M^2 - \mu^2})^r \\ &= \frac{\lambda^{r+1}(m_0^2, m_1^2, m_2^2) \Gamma(\frac{r}{2} + 1) \sqrt{\pi}}{2m_0 \Gamma(\frac{r}{2} + \frac{3}{2})}. \end{aligned}$$

The recurrence relation becomes much more interesting for the three-body case,

$$4\pi^2 \rho_{m_0 \rightarrow m_1+m_2+m_3}^{2\ell} = \iiint d\mu_1 d\mu_2 d\mu_3 \delta(\mu_1 + \mu_2 + \mu_3) \rho_{m_0 \rightarrow m_{1\mu_1}+m_{2\mu_2}+m_{3\mu_3}}^{2\ell-1} \tag{8}$$

upon recalling that even- and odd-dimensional phase space behave rather differently: odd  $D$  leads to a Laurent series in the masses, while even  $D$  generally leads to elliptic functions [3–5]; equation (8) signifies that there exists an integral relation between elliptic functions and polynomials/poles. Moreover, the nature of the integral displays *explicit* symmetry in the masses which is useful. Because of the  $\delta$  function constraint and the fact that the  $\mu$  run over positive and negative values, the rhs of (8) can be broken up into the sum of three terms:

$$4\pi^2 \rho_{m_0 \rightarrow m_1+m_2+m_3}^{2\ell} = 2 \int_0 \int_0 d\mu_1 d\mu_2 \rho_{m_0 \rightarrow m_{1\mu_1}+m_{2\mu_2}+m_{3\mu_1+\mu_2}}^{2\ell-1} + 2 \text{ cyclic perms.}$$

We may quickly check the truth of relation (8) for the test case  $\ell = 2$  when all particles are massless, since that limit of phase space simply yields

$$\rho_{m_0 \rightarrow 0+0+0}^4 = \lim_{\ell \rightarrow 2} \frac{(4\pi)^{1-2\ell} [\Gamma(\ell - 1)]^3 m_0^{4\ell-6}}{2\Gamma(3\ell - 3)\Gamma(2\ell - 2)} = \frac{m_0^2}{256\pi^3}$$

for the lhs. On the other hand, for the rhs, substituting the 3D result [6],

$$\rho_{m_0 \rightarrow m_1+m_2+m_3}^3 = (m_0 - m_1 - m_2 - m_3)\theta(m_0 - m_1 - m_2 - m_3)/16\pi m_0$$

each of the three permutations produces the same answer and we obtain a perfect check of the recurrence relation. However, one learns something new in the massive case since a *new* symmetrical representation for 4D phase space emerges:

$$\rho_{m_0 \rightarrow m_1+m_2+m_3}^4 = \frac{1}{32\pi^3 m_0} \int_0 \int_0 d\mu_1 d\mu_2 (m_0 - m_{1\mu_1} - m_{2\mu_2} - m_{3(\mu_1+\mu_2)}) + 2 \text{ other perms.} \tag{9}$$

One expects the right-hand integral to produce elliptic functions, but the main virtue of (9) is that we get a pleasingly symmetrical sum of them which in principle ought to match an earlier form [7], albeit obtained in a different manner. Although we have not yet succeeded in establishing the precise relation with the Jacobian zeta function form, we have performed a series of *successful* numerical checks of (9), using typical mass values.

The same thing happens for larger  $N$ . For instance in four-body decay there arise four permutations with three of the  $\mu_i$  having the same sign, opposite to the last one, plus six permutations where two pairs of  $\mu_i$  have the same sign and opposite to the other pair. This would provide an elegant symmetrical way of evaluating four-body decay without resorting to Almgren’s nonsymmetrical way [4] of pairing two bodies together and then summing over their pairwise mass sums.

One may of course continue in this vein and discuss spaces with the topology  $M^{D-n} \otimes T^n$ , but one learns very little new by this ruse because the process just yields a set of angles  $\theta_j = y_j/R_j$  and a set of mode numbers  $k_j$  which collectively lead to  $m_{\mathbf{k}}^2 = m^2 + \sum_j (k_j/R_j)^2$ . There is little gain in taking the limit as each  $R_j \rightarrow \infty$  separately because we only care for the final result where none of the radii is finite.

### 3. Relations between $D$ and $D - 2$ phase space

Next we shall suppose that we are dealing with the direct topology  $M^{D-2} \otimes S^2$  so that the two extra angular coordinates are confined to the surface of a 2-sphere having radius  $R$ ; thus  $\vec{y} = R(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ , whereas previously  $y$  was identified with the circumference  $R\theta$  rather than the radius vector. In such a situation expand the fields in spherical harmonics,

$$\Phi(\mathbf{x}, \hat{y}) = \sum_{j,k} \phi_{jk}(\mathbf{x}) Y_{jk}(\theta, \varphi) / R \quad (10)$$

before considering the limit of large  $R$ . The chosen factors ensure that the lower-dimensional field modes  $\phi$  are properly normalized:

$$S_{\text{free}} = \frac{1}{2} \int d^D x [(\partial_x \Phi)^2 - m^2 \Phi^2] = \sum_{j,k} \frac{1}{2} \int d^{D-2} \mathbf{x} [(\partial_x \phi_{jk}^*)(\partial_x \phi_{jk}) - m_j^2 \phi_{jk}^* \phi_{jk}] \quad (11)$$

where  $m_j^2 = m^2 + j(j+1)/R^2$ . (In equation (11) it is really meant that  $d^D x = d^{D-2} \mathbf{x} R^2 d^2 \Omega$  and  $(\partial \Phi)^2 = g^{ab} \partial_a \Phi \partial_b \Phi$ .) Again we note that  $\phi_{jk} = \phi_{j-k}^*$ , so the complex modes are found by just summing over positive  $k$  and discarding the factor of  $1/2$ .

The two-body recurrence relation can be verified in its entirety, since the effective coupling of a zero-mode mass  $m_0$  field  $\Sigma = \sigma/\sqrt{4\pi}R$  to two others yields a uniform amplitude, independently of the angular momentum eigenvalues  $j, k$  as we see from

$$\int d^D x \Phi_1 \Phi_2 \Sigma = \sum_{j,k} \int d^{D-2} \mathbf{x} \sigma \Phi_{1jk} \Phi_{2j-k} / \sqrt{4\pi} R. \quad (12)$$

The  $(2j+1)$  degeneracy in  $k$  leads to

$$\rho_{m_0 \rightarrow m_1+m_2}^D = \frac{1}{4\pi R^2} \sum_j (2j+1) \rho_{m_0 \rightarrow m_{1j}+m_{2j}}^{D-2} \quad m_{ij}^2 = m_i^2 + \left(\frac{j}{R}\right)^2. \quad (13)$$

In the limit  $(R, j) \rightarrow \infty$ , with  $\mu = j/R$ , this reduces to the continuum prediction,

$$\rho_{m_0 \rightarrow m_1+m_2}^D = \int_0^\infty d\mu^2 \rho_{m_0 \rightarrow m_{1\mu}+m_{2\mu}}^{D-2} / 4\pi \quad (14)$$

which is readily verified from the explicit result (6).

The three-body case is altogether more fascinating. Here the effective interaction reduces to

$$\begin{aligned} \int d^D x \Phi_1 \Phi_2 \Phi_3 \Sigma &= \frac{1}{\sqrt{4\pi} R^2} \sum_{j_1, k_1} \sum_{j_2, k_2} \sum_{j_3, k_3} \int d^{D-2} \mathbf{x} \sigma \Phi_{1j_1 k_1} \Phi_{2j_2 k_2} \Phi_{3j_3 k_3} \\ &\times \int d^2 \Omega Y_{j_1 k_1} Y_{j_2 k_2} Y_{j_3 k_3}. \end{aligned} \quad (15)$$

To make progress, use the orthogonality property of spherical harmonics [8]

$$\int d^2 \Omega Y_{j_1 k_1} Y_{j_2 k_2} Y_{j_3 k_3} = \sqrt{\frac{(2j_1+1)(2j_2+1)(2j_3+1)}{4\pi}} \begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{pmatrix}.$$

This then specifies the magnitudes of the mode couplings. When evaluating the sum over modes, apply the completeness relation of C-G coefficients,

$$\sum_{k_2, k_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{pmatrix} \begin{pmatrix} j_4 & j_2 & j_3 \\ k_4 & k_2 & k_3 \end{pmatrix} = \frac{\delta_{j_1 j_4} \delta_{k_1 k_4}}{2j_1 + 1}$$

signifying

$$\sum_{k_1, k_2, k_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{pmatrix}^2 = \sum_{k_1} \frac{1}{2j_1 + 1} = 1.$$

Therefore, the summation over modes produces the discrete recurrence relation,

$$\rho_{m_0 \rightarrow m_1+m_2+m_3}^D = \sum_{j_1, j_2, j_3} \frac{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)}{(4\pi R^2)^2} \begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix}^2 \rho_{m_0 \rightarrow m_1 j_1 + m_2 j_2 + m_3 j_3}^{D-2} \quad (16)$$

whose continuum limit is of interest. To take  $R \rightarrow \infty$ , first note that Wigner’s 3-j symbol is [8]

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} \equiv (-1)^{-(j_1+j_2+j_3)/2} \frac{(\frac{j_1+j_2+j_3}{2})! \sqrt{(-j_1+j_2+j_3)!(j_1-j_2+j_3)!(j_1+j_2-j_3)!}}{\sqrt{(1+j_1+j_2+j_3)!(-\frac{j_1+j_2+j_3}{2})!(\frac{j_1-j_2+j_3}{2})!(\frac{j_1+j_2-j_3}{2})!}}.$$

Since we are concerned with the large  $j$  limit, apply Stirling’s formula,

$$\frac{a!}{(a/2)!^2} \simeq \frac{2^{a+1/2}}{\sqrt{\pi a}}$$

to the previous expression. The square of the 3-j symbol then magically simplifies to the inverse area of a triangle having sides  $j_1, j_2, j_3$ :

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix}^2 \simeq \frac{2\theta(\lambda_E)}{\pi \lambda_E (j_1^2, j_2^2, j_3^2)} \quad \lambda_E^2 \equiv -\lambda^2 \quad (17)$$

which makes good sense, recalling the vector addition formula for angular momenta. One finishes with the continuum result

$$\rho_{m_0 \rightarrow m_1+m_2+m_3}^D = \frac{1}{4\pi^3} \int_0 \int_0 \int_0 d\mu_1 d\mu_2 d\mu_3 \frac{\mu_1 \mu_2 \mu_3}{\lambda_E(\mu_1^2, \mu_2^2, \mu_3^2)} \rho_{m_0 \rightarrow m_1 \mu_1 + m_2 \mu_2 + m_3 \mu_3}^{D-2}. \quad (18)$$

This recurrence is very difficult to verify in general, especially for even  $D$ . We can however make a fist of it for 5D in the massless limit when the check collapses to the veracity of

$$\rho_{m_0 \rightarrow 0+0+0}^5 = \frac{1}{64\pi^4} \int_0 \int_0 \int_0 d\mu_1 d\mu_2 d\mu_3 \frac{\mu_1 \mu_2 \mu_3 \theta(\lambda_E)}{\lambda_E(\mu_1^2, \mu_2^2, \mu_3^2)} \left(1 - \frac{\mu_1 + \mu_2 + \mu_3}{m_0}\right). \quad (19)$$

The lhs of (19) is known to be equal to  $m_0^4/53760\pi^3$ . To integrate the rhs, change variables according to  $\mu_1 = w/2 - u, \mu_2 = w/2 - v$  and  $\mu_3 = v + u$ , so,

$$\int_0 \int_0 \int_0 d\mu_1 d\mu_2 d\mu_3 X = 2 \int_0^{m_0} dw \int_0^{w/2} dv \int_0^{w/2-v} du X$$

may be carried out with  $X \propto \frac{(w/2-u)(w/2-v)(u+v)(m_0-w)}{\sqrt{uv[2w(u+v)-w^2]}}$ . The result indeed reproduces the lhs. We have also verified the truth of (19) numerically.

The miraculous birth of the Euclidean Källén function  $\lambda_E$  in (17) and (19) can be rendered less mysterious if we look upon this as the result of integrating over the last two components of momentum  $\vec{\mu}$ , corresponding to a ‘radial kernel’. Thus from the mathematical fact that

$$\left(\prod_{i=1}^3 \int \frac{d^2 \vec{\mu}_i}{(2\pi)^2}\right) (2\pi)^2 \delta^2\left(\sum_{i=1}^3 \vec{\mu}_i\right) F(|\vec{\mu}_j|) = \left(\prod_{i=1}^3 \int \frac{\mu_i d\mu_i d\theta_i}{(2\pi)^2}\right) F(\mu_j) \int d^2 \vec{k} e^{i\vec{k} \cdot \sum \vec{\mu}_i}$$

we find that this expression equals [9, 10]

$$\begin{aligned} &\left(\prod_{i=1}^3 \int \frac{\mu_i d\mu_i}{2\pi}\right) F(\mu_j) 2\pi \int_0^\infty J_0(k\mu_1) J_0(k\mu_2) J_0(k\mu_3) k dk \\ &= \left(\prod_{i=1}^3 \int \frac{\mu_i d\mu_i}{2\pi}\right) \frac{4\theta(\lambda_E) F(\mu_j)}{\lambda_E (\mu_1^2, \mu_2^2, \mu_3^2)}. \end{aligned}$$

Moreover, the extension to  $N$ -body phase space suggests itself immediately via an  $N$ -fold kernel, associated with the integral  $2\pi \int_0^\infty (\prod_{i=1}^N J_0(k\mu_i)) k dk$  although this is not readily stated in terms of simple functions for  $N > 4$ ; something geometrical associated with the lengths  $\mu_i$  is certainly involved. This kernel has to be folded over  $\rho_{m_0 \rightarrow m_1\mu_1 + \dots + m_N\mu_N}^{D-2}$  and integrated with respect to all  $\mu_i d\mu_i$  to establish the recurrence.

**4. Relations between  $D$  and  $D - n$  phase space**

With the torus and 2-sphere thoroughly understood, it is natural to extend the argument to coordinates confined to an  $n$ -sphere where the topology is  $M^{D-n} \otimes S^n$ . Here we need to make use of hyperspherical harmonics [11] defined over  $n$  angles. Associated with them are the generalized quadratic Casimir operator with eigenvalue  $j(j + n - 1)$  and  $n - 1$  angular momentum components (generically labelled  $\mathbf{k}$ ), producing a degeneracy of  $h_{jn} = (2j + n - 1) \cdot (j + n - 2)! / j!(n - 1)!$  Thus the free action may be normalized according to equation (11), where

$$\Phi(\mathbf{x}, \hat{y}) = \sum_{j, \mathbf{k}} \phi_{j, \mathbf{k}}(\mathbf{x}) Y_{j, \mathbf{k}}(\hat{y}) / R^{n/2}$$

and we must sum over  $(n - 1)$  of the  $\mathbf{k}$  labels. Now the squared mass equals  $m^2 + j(j + n - 1) / R^2$  of course, because the last term corresponds to a hyperspherical Laplacian eigenvalue [11].

Running through similar steps as before and skipping details, we arrive at the two-body recurrence relation for finite  $R$ ,

$$\rho_{m_0 \rightarrow m_1+m_2}^D = \sum_j h_{jn} \rho_{m_0 \rightarrow m_1\mu_1+m_2\mu_2}^{D-n} / \Omega_n R^n \tag{20}$$

where  $\Omega_n = 2\pi^{(n+1)/2} / \Gamma((n + 1)/2)$  is the total solid angle corresponding to  $n$  angular coordinates. In the limit of large  $R$  and therefore large  $j$ , since  $h_{jn} \simeq 2j^{n-1} / \Gamma(n)$ , we obtain the continuum limit,

$$\rho_{m_0 \rightarrow m_1+m_2}^D = \frac{\Gamma((n + 1)/2)}{\pi^{(n+1)/2} \Gamma(n)} \int_0^\infty d\mu \mu^{n-1} \rho_{m_0 \rightarrow m_1\mu+m_2\mu}^{D-n} \tag{21}$$

and this is readily checked via the explicit answer (6). Equations (7) and (14) are particular cases of (21).

While we have succeeded in treating the two-body decay by this procedure, it clearly becomes unwieldy and probably useless for  $N > 2$  and larger values of  $n$ , since we would have to integrate over multiproducts of hyperspherical harmonics, which are not exactly well known, though some fancy generalizations of 3-j symbols and the like must exist. On the



other hand, one can make much better progress by regarding the recurrence as the result of integrating over the last  $n$  momenta  $\vec{\mu}$ :

$$\rho_{m_0 \rightarrow m_1 + \dots + m_N}^D = \left( \prod_{i=1}^N \int \frac{d^n \vec{\mu}_i}{(2\pi)^n} \right) \int d^n \vec{k} e^{i\vec{k} \cdot \sum_i \vec{\mu}_i} \rho_{m_0 \rightarrow m_{1\mu_1} + \dots + m_{N\mu_N}}^{D-n}. \tag{22}$$

Now in general [1],

$$\int d^n \vec{\mu} \exp(i\vec{k} \cdot \vec{\mu}) f(\mu) = \int_0^\infty (2\pi\mu)^{n/2} J_{n/2-1}(k\mu) f(\mu) d\mu / k^{n/2-1}$$

and  $d^n \vec{k} = k^{n-1} dk \Omega_{n-1} = 2k^{n-1} d\mu \cdot \pi^{n/2} / \Gamma(n/2)$ , so (20) simplifies to

$$\begin{aligned} \rho_{m_0 \rightarrow m_1 + \dots + m_N}^D &= \left( \prod_{i=1}^N \int_0^\infty \frac{\mu_i^{n/2} d\mu_i}{(2\pi)^{n/2}} \right) \int_0^\infty \frac{2\pi^{n/2} k^{n-1} dk}{\Gamma(n/2)} \left( \prod_{i=1}^N \frac{J_{n/2-1}(k\mu_i)}{k^{n/2-1}} \right) \\ &\times \rho_{m_0 \rightarrow m_{1\mu_1} + \dots + m_{N\mu_N}}^{D-n}. \end{aligned} \tag{23}$$

As far as we are aware there is no amenable formula for the radial kernel: an integral over the product of  $N$  Bessel functions of the first kind with different arguments for all  $N$ . However, the cases  $N = 2$  and  $N = 3$  are known and for positive  $a, b, c$ , read [9]

$$\begin{aligned} \int_0^\infty J_\nu(ax) J_\nu(bx) x dx &= 2\delta(a^2 - b^2) \\ \int_0^\infty J_\nu(ax) J_\nu(bx) J_\nu(cx) x^{1-\nu} dx &= 2\theta(\lambda_E) \lambda_E^{2\nu-1} / (8abc)^\nu \Gamma(\nu + 1/2) \sqrt{\pi}. \end{aligned}$$

For  $N = 4$  it is even known that [10]

$$\int_0^\infty J_0(ax) J_0(bx) J_0(cx) J_0(dx) x dx = \mathbf{K}(\sqrt{abcd}/\Delta) / \pi^2 \Delta$$

where  $16\Delta^2 = (a + b + c - d)(b + c + d - a)(c + d + a - b)(d + a + b - c)$  is associated with the maximal area of a (cocyclic) quadrilateral formed by the lengths  $a, b, c, d$  and  $\mathbf{K}$  is the complete elliptic integral of the first kind. However, this means that we may at least write a simple closed form for the three-body recurrence relation, when  $n$  is arbitrary:

$$\rho_{m_0 \rightarrow m_1 + m_2 + m_3}^D = \left( \prod_{i=1}^3 \int_0^\infty \frac{\mu_i d\mu_i}{(2\pi)^{n/2}} \right) \frac{2^{3-n/2} \theta(\lambda_E) \pi^{n/2-1}}{\lambda_E^{3-n} (\mu^2, \mu_2^2, \mu_3^2) \Gamma(n-1)} \rho_{m_0 \rightarrow m_{1\mu_1} + m_{2\mu_2} + m_{3\mu_3}}^{D-n}. \tag{24}$$

An interesting case occurs when  $N = n = 3$ , whereupon (24) reduces to

$$\rho_{m_0 \rightarrow m_1 + m_2 + m_3}^D = \int_0^\infty \int_0^\infty \int_0^\infty d\mu_1^2 d\mu_2^2 d\mu_3^2 \frac{\theta(\lambda_E)}{64\pi^4} \rho_{m_0 \rightarrow m_{1\mu_1} + m_{2\mu_2} + m_{3\mu_3}}^{D-3} \tag{25}$$

and because 3D phase space is so simple [6], we obtain an intriguing representation for 6D phase space on setting  $D = 6$ . Specifically,

$$\rho_{m_0 \rightarrow m_1 + m_2 + m_3}^6 = \int_0^\infty da \int_0^\infty db \int_0^\infty dc \frac{\theta(\lambda_E(a, b, c))}{64\pi^4} \frac{m_0 - \sqrt{m_1^2 + a} - \sqrt{m_2^2 + b} - \sqrt{m_3^2 + c}}{16\pi m_0}.$$

With larger  $N$ , presumably the radial kernel on the right-hand side of (23) involves all lengths  $\mu_i$  and something geometrically more complicated than  $\lambda_E$ , connected with the closed figure  $\sum_i \vec{\mu}_i = 0$ . (Thus we expect it to vanish when any length exceeds the lengths of the other three, amongst other conditions. This is a very interesting topic worth future investigation.)

## 5. Conclusions

In this paper we have established the connection between relativistic phase space for different dimensions by two methods. They complement Almgren's connection between phase space in the same dimension, but for different numbers of decay products. Our recurrence relations have practical utility when the difference in dimensions  $n$  is odd since it leads to elegant symmetrical representations of  $\rho$  for even  $D$  involving elliptic functions.

Last but not least, one should observe that phase space is nothing but the imaginary part of a sunset Feynman diagram. Since recurrence relations between sunset diagrams differing in dimensionality by 2 have been found by other methods [12], it should be possible to rewrite our results that way, for even  $n$  at any rate. We should also point out that as a rule those relations do not connect even and odd  $D$  because they are obtained by integration by parts from scalar particle Feynman graphs.

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See also equation (25) in Davydychev A I and Tausk J B 1996 *Phys. Rev.* **53** 7381